ISRAEL JOURNAL OF MATHEMATICS **106** (1998), 285–311

$\mathcal{M}$-GAP CONJECTURE AND m-NORMAL THEORIES

BY

LUDOMIR NEWELSKI*

Mathematical Institute, University of Wroclaw
pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland
e-mail: newelski@math.uni.wroc.pl
and
Mathematical Institute of the Polish Academy of Sciences

ABSTRACT

We find a small weakly minimal theory with an isolated weakly minimal type of $\mathcal{M}$-rank ∞ and an isolated weakly minimal type of arbitrarily large finite $\mathcal{M}$-rank. These examples lead to the notion of an m-normal theory. We prove the $\mathcal{M}$-gap conjecture for m-normal T. In superstable theories with few countable models we characterize traces of complete types as traces of some formulas. We prove that a 1-based theory with few countable models is m-normal. We investigate generic subgroups of small superstable groups. We compare the notions of independence induced by measure (μ-independence) and category (m-independence).

0. Introduction

Throughout, unless we say otherwise, T is a small stable theory in a language L, and we work within a monster model $\mathfrak{C} = \mathfrak{C}^{eq}$ of T. The main feature that distinguishes small superstable theories from ω-stable ones is the presence of complete types over finite sets, with infinite multiplicity. Indeed, [M, A.16] proves that a small superstable theory T is ω-stable iff for every finite set $A \subset \mathfrak{C}$ and $p \in S(A)$, p has finite multiplicity, that is the set of stationarizations of p (over $acl(A)$ or over $\mathfrak{C}$) is finite. Hence, after Vaught's conjecture for ω-stable theories was proved [SHM], investigation of the ways in which multiplicity of p may be infinite

* Research supported by KBN grant 2 P03A 006 09.
Received October 6, 1996 and in revised form February 19, 1997

seems a necessary step towards a proof of Vaught's conjecture for superstable theories. This was the motivation for a number of papers that I have written in recent years [Ne1, Ne3, Ne4, Ne5, Ne7]. When speaking about stationarizations of types, we use the following notions.

If $s(x)$ is a (partial) type over $\mathfrak{C}$, then $[s]$ is the class of types over $\mathfrak{C}$, containing s. Let $A \subset \mathfrak{C}$ be finite. So sets of the form $S(A) \cap [\varphi]$, where $\varphi(x)$ is a formula with parameters from A, are a basis of the topology on $S(A)$. We identify strong types over A with types in $S(acl(A))$.

The trace of s over A is the set $\mathrm{Tr}_A(s) = \{p(x) \in S(acl(A))\colon p(x) \cup s(x)$ is consistent$\}$. $\mathrm{Tr}_A(a/B)$ denotes $\mathrm{Tr}_A(tp(a/B))$ and $\mathrm{Tr}(s)$ denotes $\mathrm{Tr}_\emptyset(s)$. In particular, if $p \in S(A)$ then $\mathrm{Tr}_A(p)$ is the set of stationarizations of p over A. When $A \subset B$, $S_{p,nf}(B)$ denotes the set of non-forking extensions of p in $S(B)$. Notice that $\mathrm{Tr}_A(s)$ is a closed subset of $S(acl(A))$. Hence for $p \in S(A)$ it is reasonable to investigate the 'topological shape' of $\mathrm{Tr}_A(p)$, rather than merely count the number of elements of it. The first result in this direction [Ne1] states that

(*) if T has few (that is $< 2^{\aleph_0}$) countable models and $p \in S(A)$ is weakly minimal then either $\mathrm{Tr}_A(p)$ is finite or $\mathrm{Tr}_A(p)$ is open (that is, p is isolated).

This fact (called Saffe's condition) was decisive in the proof of Vaught's conjecture for weakly minimal theories [Bu1]. Saffe's condition fails in general, for arbitrary p, at least when stated in the above way. So in order to measure the size of the set of stationarizations of $p \in S(A)$, I introduced a multiplicity rank $\mathcal{M}$ defined by the following conditions.

(1) $\mathcal{M}(a/A) \geq 0$.

(2) $\mathcal{M}(a/A) \geq \alpha + 1$ iff for some finite $B \supset A$ with $a \mathop{\smile\!\!\!\!\!\mid} B(A)$, $\mathcal{M}(a/B) \geq \alpha$ and $\mathrm{Tr}_A(a/B)$ is nowhere dense in $\mathrm{Tr}_A(a/A)$.

(3) $\mathcal{M}(a/A) \geq \delta$ for limit δ iff $\mathcal{M}(a/A) \geq \alpha$ for every $\alpha < \delta$.

Regarding this definition notice that for any $A \subseteq B \subseteq \mathfrak{C}$ and $a \in \mathfrak{C}$ the following conditions are equivalent.

- $\mathrm{Tr}_A(a/B)$ is nowhere dense in $\mathrm{Tr}_A(a/A)$.
- $\mathrm{Tr}_A(a/B)$ is meager in $\mathrm{Tr}_A(a/A)$.
- $\mathrm{Tr}_A(a/B)$ has empty interior in $\mathrm{Tr}_A(a/A)$.
- $\mathrm{Tr}_A(a/B)$ is not open in $\mathrm{Tr}_A(a/A)$.

We see that for $a \in \mathfrak{C}, \mathcal{M}(a/A)$ depends only on $tp(a/A)$, so for $p \in S(A)$ we can define $\mathcal{M}(p)$ as $\mathcal{M}(a/A)$ for any a realizing p. $\mathcal{M}(a)$ denotes $\mathcal{M}(a/\emptyset)$.

We say that T is m-stable if $\mathcal{M}(p) < \infty$ for any p (over a finite set). In [Ne4, Ne7] it is proved that if T is superstable and has few countable models, then $\mathcal{M}(p)$ is finite for any p (over a finite set). On the other hand, in a small superstable T, using $\mathcal{M}$-rank we can produce certain locally modular types (called meager types) in a mechanical way, just like U-rank yields regular types (see [Ne4, Ne5]). Unfortunately, there are no examples of a type p in a small T with $\omega \leq \mathcal{M}(p) < \infty$. In [Ne7] I formulated the following conjecture.

$\mathcal{M}$-GAP CONJECTURE: *In a small superstable T there are no p with $\omega \leq \mathcal{M}(p) < \infty$.*

Were this conjecture true, a large part of [Ne5] would be redundant, since it deals with types of infinite $\mathcal{M}$-rank. The status of this conjecture is unclear even in the simplest case of a weakly minimal T. This is the main motivation for the present paper.

Below we construct small weakly minimal groups with an isolated generic type of arbitrarily large finite $\mathcal{M}$-rank, and of $\mathcal{M}$-rank ∞. The next sections are devoted to explaining why finding a counterexample to the $\mathcal{M}$-gap conjecture may be hard. Specifically, the examples below are m-normal (for definition see the end of this section). We prove that the $\mathcal{M}$-gap conjecture is true for m-normal theories. Also we study local properties of $\mathcal{M}$-rank and m-independence (defined below), and investigate generic subgroups of type-definable groups. We prove that if T has few countable models then traces of complete types are traces of some formulas. As a consequence we get that a 1-based theory with few countable models is m-normal. In the last section we discuss the properties of the set of generic types of a group G; also we comment there on a notion of independence induced by measure and compare it with the notion of m-independence defined below (induced by category).

The notion of multiplical independence (m-independence, for short) is defined in terms of traces of types; we denote it by $\overset{m}{\perp}$. It refines (for small stable T) the notion of independence. We say that

> a is **m-independent** from b over c (symbolically: $a\overset{m}{\perp}b(c)$) if $a\perp b(c)$ and $\mathrm{Tr}_c(a/bc)$ is open in $\mathrm{Tr}_c(a/c)$.

Below we collect the basic properties of m-independence (the proofs may be found in [Ne3, Ne5]).

LEMMA 0.1:

(1) *(symmetry) $a\overset{m}{\perp}b(A)$ implies $b\overset{m}{\perp}a(A)$.*

(2) *(transitivity)* $ab \overset{m}{\perp} c(A)$ *iff* $a \overset{m}{\perp} c(Ab)$ *and* $b \overset{m}{\perp} c(A)$.
(3) $\mathcal{M}(a/A) \leq \mathcal{M}(ab/A) \leq \mathcal{M}(a/Ab) \oplus \mathcal{M}(b/A)$.
(4) *If* $a \perp b(A)$ *then* $\mathcal{M}(a/Ab) + \mathcal{M}(b/A) \leq \mathcal{M}(ab/A)$.
(5) *If* $a \overset{m}{\perp} b(A)$ *then* $\mathcal{M}(ab/A) = \mathcal{M}(a/A) \oplus \mathcal{M}(b/A)$ *and* $\mathcal{M}(a/A) = \mathcal{M}(a/Ab)$.
(6) *If* $\mathcal{M}(a/A) < \infty$ *and* $a \perp b(A)$ *then* $\mathcal{M}(a/A) = \mathcal{M}(a/Ab)$ *implies* $a \overset{m}{\perp} b(A)$.
(7) *If* $B \supset A$ *is finite then for every* a *there is an* $a' \equiv a(A)$ *with* $a' \overset{m}{\perp} B(A)$.
(8) *Assume* $A \subset B \subset C$. *Then either* $\mathrm{Tr}_A(a/C)$ *is open in* $\mathrm{Tr}_A(a/B)$ *or* $\mathrm{Tr}_A(a/C)$ *is nowhere dense in* $\mathrm{Tr}_A(a/B)$.

Now we construct small weakly minimal groups with a generic type of arbitrarily large finite $\mathcal{M}$-rank, and of $\mathcal{M}$-rank ∞.

Let $V = {}^{\omega\times\omega}2 = \prod_n {}^{\{n\}\times\omega}2$, equipped with the usual product topology. V can be made into a group structure, with pointwise addition modulo 2. Let $P_n = \{f \in V\colon f|(n \times n) \equiv 0\}$ and $G = (V, +, P_n)_{n<\omega}$. G is the standard example of a small weakly minimal group. Also, $Th(G)$ has few countable models, hence necessarily $\mathcal{M}$-rank of any isolated generic type is 1 (by [Ne1]). The set of generic types of G is naturally homeomorphic with $\prod_n {}^{\{n\}\times\omega}2$. Let

$$H_n = \prod_{i<n}\{0\} \times \prod_{i\geq n} {}^{\{i\}\times\omega}2 = \{f \in V\colon f|(n \times \omega) \equiv 0\}.$$

We see that $V = H_0 > H_1 > H_2 > \ldots$, moreover every H_{i+1} is closed and nowhere dense in H_i. Our intention is to expand G so that it remains small and weakly minimal, but $H_i, i < \omega$, are type-definable over $\emptyset$ generic subgroups of G. We cannot simply expand G by adding the H_i's as the new predicates, since this would destroy not only weak minimality, but superstability of T. Instead we name in G certain generic clopen subgroups, which approximate the H_i's. Specifically, let $H_{n,m} = \{f \in V\colon f|(n \times m) \equiv 0\}$. Notice that $P_n = H_{n,n}$. We see that $H_n = \bigcap_m H_{n,m}$. Let $G_\infty = (V, +, \{P_n, n < \omega\}, \{H_{n,m}\colon n < m < \omega\})$. G_∞ is weakly minimal and small.

For instance, we describe $S(\emptyset)$. For generic $a \in G_\infty$ we define a function $f_a\colon \{(n,m)\colon n \leq m\} \to \{0,1\}$ by $f_a(n,m) = 0$ iff $a \in H_{n,m}$. Clearly f_a determines $tp(a/\emptyset)$. We show that the set $F = \{f_a\colon a \in G_\infty \text{ is generic }\}$ is countable.

Suppose $f \in F$ and $f \not\equiv 0$. Let (n_0, m_0) be minimal (with respect to the lexicographical order on $\omega \times \omega$) such that $m_0 \geq n_0$ and $f(n_0, m_0) = 1$. It follows that for any n, m with $m \geq n$:
– if $n < n_0$ then $f(n,m) = 0$,
– if $n = n_0$ and $m < m_0$ then $f(n,m) = 0$,
– if $n \geq n_0$ and $m \geq n_0$ then $f(n,m) = 1$.

There are yet finitely many pairs (n, m) with $m \geq n$ to consider, namely these with $m_0 > m \geq n \geq n_0$. This shows that F is countable.

Actually, $max\{CB(p): p \in S(\emptyset)\} = \omega$ here. Also by the choice of H_i's, $\mathcal{M}$-rank of any isolated generic type over $\emptyset$ in G_∞ equals ∞. If we define G_N (for some $N > 0$) as $(V, +, \{P_n, n < \omega\}, \{H_{n,m}, n < m < \omega \text{ and } n < N\})$, then we get that G_N is weakly minimal, small and the $\mathcal{M}$-rank of any isolated generic type over $\emptyset$ in G_N equals N.

> We say that T is **m-normal** if for all finite sets $A \subset B$ and $a \perp B(A)$, there is an $E \in FE(A)$ (in suitable variables) such that the orbit of $\mathrm{Tr}_A(a/B) \cap [E(x, a)]$ under the action of $Aut(\mathfrak{C}/Aa)$ is finite.

The following remark yields an equivalent definition.

Remark 0.2: $\mathrm{Tr}_A(a/B) \cap [E(x, a)]$ has finitely many conjugates over Aa iff for some $b \in acl(A)$, $\mathrm{Tr}_A(a/Bb)$ is $Aut(\mathfrak{C}/Aab)$-invariant.

Proof: $\rightarrow$. First notice that $\mathrm{Tr}_A(a/B) \cap [E(x, a)] = \mathrm{Tr}_A(a/Ba')$, where $a' = a/E$. Let $X_0 = \mathrm{Tr}_A(a/Ba'), X_1, \ldots, X_n$ (for some n) be the Aa-conjugates of $\mathrm{Tr}_A(a/Ba')$. For $i > 0, X_i \neq X_0$, hence there is $a_i \in acl(A)$, a name of an equivalence class of some $E_i \in FE(A)$ meeting X_0 and disjoint from X_i. Let a'' be the concatenation of $a_1, \ldots, a_n$. We see that $a'' \in acl(A)$ and $\mathrm{Tr}_A(a/Ba')$ is $Aut(\mathfrak{C}/Aaa'')$-invariant. $\mathrm{Tr}_A(a/Ba'a'')$ is open in $\mathrm{Tr}_A(a/Ba')$, hence there is $E' \in FE(A)$ such that $\mathrm{Tr}_A(a/Ba'a'') \cap [E'(x, a)] = \mathrm{Tr}_A(a/Ba') \cap [E'(x, a)]$, and this set is clearly $Aut(\mathfrak{C}/Aaa'')$-invariant. Let $b = a'a''(a/E')$. We see that $\mathrm{Tr}_A(a/Bb) = \mathrm{Tr}_A(a/Ba'a'') \cap [E'(x, a)]$ is $Aut(\mathfrak{C}/Aab)$-invariant.

$\leftarrow$. As above, $\mathrm{Tr}_A(a/Bb)$ is open in $\mathrm{Tr}_A(a/B)$, hence for some $E \in FE(A)$, $\mathrm{Tr}_A(a/B) \cap [E(x, a)]$ is $Aut(\mathfrak{C}/Aab)$-invariant. As $b \in acl(A)$, we get that $\mathrm{Tr}_A(a/B) \cap [E(x, a)]$ has finitely many conjugates over Aa.

The definition of m-normality is parallel to the definition of weak normality (1-basedness). More details are given in the next section. We prove there the $\mathcal{M}$-gap conjecture for m-normal theories. One can see that the theories $Th(G_\infty), Th(G_N)$ are m-normal. In fact, for any finite set $A \subset G_\infty$ (or $\subset G_N$) and any $a \in G_\infty$ generic over A, we have that $\mathrm{Tr}(a/A)$ is a Boolean combination of generic traces of cosets of type-definable over $\emptyset$ generic subgroups of G_∞ (or G_N, respectively). Also, as the next remark shows, it suffices to check m-normality on the real sort $\mathfrak{C}_=$ of $\mathfrak{C}^{eq}$.

Remark 0.3: Assume for all finite $A \subset B \subset \mathfrak{C}$ and $a \subset \mathfrak{C}_=$ with $a \perp B(A)$, for some $b \in acl(A)$, $\mathrm{Tr}_A(a/Bb)$ is $Aut(\mathfrak{C}/Aab)$-invariant. Then T is m-normal.

Proof: Suppose a is imaginary, say a name of an E-class for some equivalence relation E. We may choose a' with $a = a'/E$ and $a' \perp^m B(Aa)$. By assumption, choose $c \in acl(A)$ with $\mathrm{Tr}_A(a'/Bc)\, Aut(\mathfrak{C}/Aa'c)$-invariant. It follows that $\mathrm{Tr}_A(a/Bc)$ is $Aut(\mathfrak{C}/Aac)$-invariant. Indeed, suppose $f \in Aut(\mathfrak{C}/ac)$ and $f(B) = B^*$. Wlog $a' \perp^m BB^*(Aa)$. This implies $B \equiv B^*(Aa'c)$. Hence $\mathrm{Tr}_A(a'/B^*c) = \mathrm{Tr}_A(a'/Bc)$ and $\mathrm{Tr}_A(a/B^*c) = \mathrm{Tr}_A(a/Bc)$.

1. m-normal theories

In this section we prove that an m-normal theory satisfies the $\mathcal{M}$-gap conjecture. We also define some special objects, called $*$-finite tuples, which play a similar role for m-independence, as imaginaries for forking independence. $*$-finite tuples allow us to re-define the notion of m-normal theory more smoothly. To justify our definition of an m-normal theory, we begin by proving that an m-stable theory is 'finitely m-based'.

PROPOSITION 1.1: *Assume T is m-stable, A, B are finite and $p \in S(A)$. Then there is a finite set C of A-independent realizations of p such that $C \perp B(A)$ and for every a realizing p with $a \perp BC(A)$, we have $a \perp^m B(AC)$.*

Proof: Suppose not. Then we can find recursively $c_n, n < \omega$, realizing p such that

(a) $C = \{c_n, n < \omega\}$ is A-independent and $B \perp C(A)$,

(b) $c_n \not\perp^m B(Ac_{<n})$ for every n.

By symmetry of m-independence, $B \not\perp^m c_n(Ac_{<n})$. This means that $\mathcal{M}(B/Ac_{<0}) > \mathcal{M}(B/Ac_{<1}) > \dots$, contradicting m-stability of T.

The next corollary says that, at least for m-stable T, $\mathcal{M}$-rank has local character.

COROLLARY 1.2: *Assume T is m-stable, $p = tp(a/A)$ for some finite A and α is an ordinal. Then the following conditions are equivalent.*

(1) $\mathcal{M}(p) \geq \alpha + 1$.

(2) *For some finite A-independent set B of realizations of p, $a \perp B(A)$, $\mathcal{M}(a/AB) \geq \alpha$ and $a \not\perp^m B(A)$.*

Proof: It suffices to prove (1) $\to$ (2). By (1) there is a finite set B' with $a \perp B'(A)$, $\mathcal{M}(a/AB') \geq \alpha$ and $a \not\perp^m B'(A)$. Let C be a finite set provided by

Proposition 1.1 (for $B := B'$). By Lemma 0.1(7) we can assume $a \overset{m}{\perp} C(AB')$. In particular, $a \perp B'C(A)$, hence by the choice of $C, a \overset{m}{\perp} B'(AC)$. Therefore

$$\mathcal{M}(a/AB') = \mathcal{M}(a/AB'C) = \mathcal{M}(a/AC).$$

Since $\mathcal{M}(a/AB') < \mathcal{M}(a/A)$, also $\mathcal{M}(a/AC) < \mathcal{M}(a/A)$. Hence $a \overset{m}{\not\perp} C(A)$. We see that $B := C$ satisfies our demands.

Corollary 1.2 implies that if T is m-stable and $\varphi(x) \in p = tp(a/A)$, then $\mathcal{M}(a/A)$ computed in $\mathfrak{C}$ as a model of T equals $\mathcal{M}(a/A)$ computed in $\varphi(\mathfrak{C})$ as a model of $T\lceil\varphi$ (or even computed in $p(\mathfrak{C})$). We can think of C in Proposition 1.1 as an 'm-basis' of B over $p(\mathfrak{C})$. The next lemma says that when T is m-normal then for a fixed a realizing p we can choose C of size 1 (just like in the case of forking independence in a 1-based theory).

LEMMA 1.3: *Assume T is m-normal, $A \subset B$ are finite and $p = tp(a/A)$ for some $a \perp B(A)$. Then there is c realizing p such that $\{B, a, c\}$ is A-independent, $a \overset{m}{\perp} B(Ac)$ and $a \overset{m}{\perp} c(AB)$.*

Proof: Choose $a' \in acl(A)$ such that $\mathrm{Tr}_A(a/Ba')$ is $Aut(\mathfrak{C}/Aaa')$-invariant. By Lemma 0.1(7) choose $c \equiv a(Ba')$ with $c \overset{m}{\perp} a(Ba')$. By symmetry (Lemma 0.1(1)) it follows that $a \overset{m}{\perp} c(B)$. This implies that

(a) $\mathrm{Tr}_A(a/Ba'c)$ is open in $\mathrm{Tr}_A(a/Ba')$.

To prove that $a \overset{m}{\perp} B(Ac)$ notice that $\mathrm{Tr}_A(a/Ba') = \mathrm{Tr}_A(c/Ba')$ is also $Aut(\mathfrak{C}/Aca')$-invariant. Hence

(b) $\mathrm{Tr}_A(a/Aca') \subset \mathrm{Tr}_A(a/Ba')$.

Since clearly $\mathrm{Tr}_A(a/Bca') \subset \mathrm{Tr}_A(a/Aca')$, by (a) we get that $\mathrm{Tr}_A(a/Bc)$ is not nowhere dense in $\mathrm{Tr}_A(a/Ac)$. By Lemma 0.1(8), $\mathrm{Tr}_A(a/Bc)$ is open in $\mathrm{Tr}_A(a/Ac)$, hence $a \overset{m}{\perp} B(Ac)$.

The next theorem says that for an m-normal theory the $\mathcal{M}$-gap conjecture is true.

THEOREM 1.4: *Assume T is small, stable and m-normal. Then there is no type p in T with $\omega \leq \mathcal{M}(p) < \infty$.*

Proof: Suppose there is a type p in T with $\omega \leq \mathcal{M}(p) < \infty$. We can assume $\mathcal{M}(p) = \omega$ and $p \in S(\emptyset)$. Let $X = \mathrm{Tr}(p)$. We define a binary relation R on X as follows. Suppose $r, r' \in X$. We have $r\,R\,r'$ iff for some a, a' realizing r, r' respectively we have $a \perp a'$ and $a \overset{m}{\not\perp} a'$. We have

(a) R is an equivalence relation.

Indeed, R is reflexive since when $r = r'$ and a, a' are independent realizations of r, then $\mathcal{M}(a/a') = 0$ and $\mathcal{M}(a/\emptyset) = \omega$, hence $a \not\perp^m a'$. R is symmetric by Lemma 0.1(1). To see that R is transitive, suppose $r\,R\,r'$ and $r'\,R\,r''$. Choose a, a', a'' realizing r, r', r'' respectively, with $a \not\perp^m a'$ and $a' \not\perp^m a''$. We can assume that a, a', a'' are independent. $a \not\perp^m a'$ gives $\mathcal{M}(a/a') < \mathcal{M}(a/\emptyset) = \omega$, hence $\mathcal{M}(a/a') < \omega$ and $\mathcal{M}(a/a'a'') < \omega$. Similarly, $\mathcal{M}(a'/a'') < \omega$. Hence by Lemma 0.1(3),

$$\mathcal{M}(a/a'') \leq \mathcal{M}(aa'/a'') \leq \mathcal{M}(a/a''a') \oplus \mathcal{M}(a'/a'') < \omega.$$

This gives $a \not\perp^m a''$ and $r\,R\,r''$.

Let $r \in X$ and let Y be the equivalence class of r. Let $Z = cl(Y)$ be the topological closure of Y. Let a realize r and let $\mathcal{P}$ be the set of non-forking extensions of p over a with finite $\mathcal{M}$-rank. We see that $Y = \bigcup\{\mathrm{Tr}(q)\colon q \in \mathcal{P}\}$. Moreover, Z is nowhere dense. Otherwise, by smallness, for some $q \in \mathcal{P}, \mathrm{Tr}(q)$ is open in Z, and open in X, a contradiction. Since Z is $Aut(\mathfrak{C}/a)$-invariant, we get that for any $r' \in Z, r\,R\,r'$, that is $Y = Z$. So we have proved that

(b) the R-equivalence class Y of r is closed and nowhere dense, and for some $q \in \mathcal{P}, \mathrm{Tr}(q)$ is open in Y.

Moreover we have

(c) for any $n < \omega$ and any open neighbourhood U of r there is $q \in \mathcal{P}$ with $\mathrm{Tr}(q) \subset U$ and $\mathcal{M}(q) > n$.

Indeed, let a realize r. Choose $E \in FE(\emptyset)$ such that $X \cap [E(x,a)] \subset U$. Let $a' = a/E$; we can add a' to the signature so that $X \subset U$. Choose a finite set B with $a \perp B, a \not\perp^m B$ and $\mathcal{M}(a/B) > n$. Clearly $\mathcal{M}(a/B)$ is finite. By Lemma 1.3 there is a c realizing p such that $\{B, a, c\}$ is independent, $a \not\perp^m B(c)$ and $a \not\perp^m c(B)$. It follows that $\mathcal{M}(a/c) = \mathcal{M}(a/B)$, and $r\,R\,stp(c)$. Switching the roles of a, c we get that there is c' realizing p with $a \perp c', a \not\perp^m c'$ and $\mathcal{M}(c'/a) > n$. So $q = tp(c'/a)$ satisfies our demands.

Now fix a type $q \in \mathcal{P}$ with $\mathrm{Tr}(q)$ open in Y, and let $n = \mathcal{M}(q)$. Let b realize q and $r' = stp(b)$. By m-normality there is $E \in FE(\emptyset)$ such that $U = \mathrm{Tr}(q) \cap [E(x,b)]$ (an open neighbourhood of r' in Y) is $Aut(\mathfrak{C}/bb')$-invariant for some $b' \in acl(\emptyset)$. It follows that

(d) for every non-forking extension q' of p over bb' with $\mathcal{M}(q')$ finite and $\mathrm{Tr}(q') \subset U$, we have $\mathcal{M}(q') \leq n$.

To see this, suppose that for some q' as in (d), $\mathcal{M}(q') > n$. Let c realize q'. Wlog a, b, c are independent. Further on we can assume that $c \overset{m}{\perp} a(bb')$. This implies $\mathcal{M}(c/abb') > n$. On the other hand, since $\mathrm{Tr}(q') \subset U$, we have that c realizes q, hence $\mathcal{M}(c/ab) \leq \mathcal{M}(c/a) = \mathcal{M}(q) = n$, a contradiction.

However, (c) holds also for r', hence we get a contradiction with (d).

In its current form, the definition of an m-normal theory does not resemble the definition of 1-based (or: weakly normal) theory. This is because we lack a counterpart of imaginary elements here. So below we introduce some other kind of objects, called $*$-finite tuples, which play for m-independence a similar role as imaginaries for forking independence.

First recall that if I is any index set and $a_I = \langle a_i, i \in I \rangle$, then Shelah considered in [Sh] the $*$-type of a_I over a set A (denoted by $tp^*(a_I/A)$) as the set of formulas $\varphi(x_J)$ over A, for some finite $J \subset I$ such that $\varphi(a_J)$ holds. Here we adopt the convention that if J is any index set then x_J is the tuple of variables $\langle x_i, i \in J \rangle$, and $a_J = \langle a_i, i \in J \rangle$. Thus $tp^*(a_I/A)$ is an element of the space of complete types over A, in variables x_I, denoted by $S_I(A)$. Topologically, it is the Stone space of the Lindenbaum–Tarski algebra of formulas in variables x_I and parameters from A. We call a_I a $*$-tuple (indexed by I).

Our aim is to regard certain $*$-tuples similarly as finite tuples of elements of $\mathfrak{C}$. All $*$-tuples form too broad a class of objects for our purposes. For instance, if $S(acl(\emptyset))$ is uncountable and a_I is an enumeration of $acl(\emptyset)$, then $S(a_I)$ is uncountable, although for any finite tuple a of elements of $\mathfrak{C}$, $S(a)$ is countable (throughout we assume T is small). So we say that

a_I is a $*$-**finite tuple** if for some finite set $A \subset \mathfrak{C}, a_I \subset dcl(A)$.

T is countable, hence if a_I is $*$-finite then we can assume I is countable. So clearly we have

Remark 1.5: If a_I is $*$-finite then $S(a_I)$ is countable.

For the purposes of forking dependence, we regard $*$-finite tuples as sets, that is $a_I \perp A(B)$ iff $\{a_i, i \in I\} \perp A(B)$. If $a_I, b_{I'}, c_{I''}$ are $*$-finite, then we write $a_I \overset{m}{\perp} b_{I'}(c_{I''})$ iff $a_I \perp b_{I'}(c_{I''})$ and $\mathrm{Tr}_{c_{I''}}(a_I/b_{I'}c_{I''})$ is open in $\mathrm{Tr}_{c_{I''}}(a_I/c_{I''})$. Here $\mathrm{Tr}_{c_{I''}}(a_I/b_{I'}c_{I''})$ is the set $\{tp(a'_I/acl(\{c_i, i \in I''\})): a'_I \equiv a_I(b_{I'}c_{I''})\}$.

Similarly we extend the definition of $\mathcal{M}$-rank $\mathcal{M}(a/A)$ to the case when a is $*$-finite and A is a finite set of $*$-finite tuples: the set B in (2) in the introduction is now required to be a finite set of $*$-finite tuples. Most importantly we have

PROPOSITION 1.6: *Lemma 0.1 remains true if a, b, c denote $*$-finite tuples and A, B, C denote finite sets of $*$-finite tuples. Also, if T is a superstable theory with few countable models then $\mathcal{M}(a/A)$ is finite for any $*$-finite tuple a and any finite set A of $*$-finite tuples.*

Proof: We concentrate on the last clause, leaving the first part of the proposition to the reader. Take a finite set A' of standard elements with $A \subseteq dcl(A')$ and $a \overset{m}{\perp} A'(A)$. Hence $\mathcal{M}(a/A') = \mathcal{M}(a/A)$, and we can assume that A is a finite set of standard elements. Next take a standard a' with $a \subseteq dcl(a')$. So $\mathcal{M}(a/Aa') = 0$ and by the few models assumption $\mathcal{M}(a'/A) < \omega$. By Lemma 0.1 we have

$$\mathcal{M}(a/A) \leq \mathcal{M}(aa'/A) \leq \mathcal{M}(a/Aa') \oplus \mathcal{M}(a'/A) < \omega.$$

From now on in this section A, B, C usually denote finite sets of $*$-finite tuples and a, b, c denote $*$-finite tuples. If we want to stress that a is a finite tuple of elements of $\mathfrak{C}$, we call a standard. We say that a_I is $*$-algebraic over A ($a_I \in acl^*(A)$) if $\{a_i, i \in I\} \subset acl(A)$. a_I is $*$-algebraic if a_I is $*$-algebraic over $\emptyset$. We say that a is algebraic over A ($a \in acl(A)$) if there are only finitely many a' such that $a' \equiv a(A)$. Similarly we define the notions of $*$-definability (dcl^*) and definability (dcl) of $*$-finite tuples. Notice that $a \in acl^*(A)$ does not imply $a \in acl(A)$, while $a \in dcl^*(A)$ implies $a \in dcl(A)$. Also, for $a \in acl^*(A)$ we have $\mathcal{M}(a/A) = 0$ iff $a \in acl(A)$. We say that $tp^*(a/A)$ is stationary if for any $a' \equiv a(A)$ we have $a' \equiv a(acl(A))$.

We should remark here that, despite our having changed the general set-up, for standard a and A, $\mathcal{M}(a/A)$ is the same, no matter if in the process of computation we use $*$-finite tuples or not.

Fact 1.7: If a, A are standard, then $\mathcal{M}(a/A) \geq \alpha + 1$ iff for some standard finite $B \supset A$, $a \perp B(A), \mathcal{M}(a/B) \geq \alpha$ and $a \overset{m}{\not\perp} B(A)$.

Proof: It suffices to prove $\rightarrow$. Suppose there is a finite set $B' \supset A$ of $*$-finite tuples such that $a \perp B'(A), \mathcal{M}(a/B') \geq \alpha$ and $a \overset{m}{\not\perp} B'(A)$. Choose a finite standard B such that $B' \subset dcl(B)$. Wlog $a \overset{m}{\perp} B(B')$. It follows that $a \perp B(A)$ and $\mathcal{M}(a/B) = \mathcal{M}(a/B') \geq \alpha$. Also, $B' \subset dcl(B)$ implies $a \overset{m}{\not\perp} B(A)$.

The next remark shows that for an ω-stable T $*$-finite tuples are redundant, and for a superstable T any $*$-finite tuple is $*$-algebraic over some standard subtuple.

Remark 1.8: (1) If T is ω-stable then every $*$-finite tuple b is equidefinable with a standard $b' \subset b$.

(2) If T is superstable, then every $*$-finite tuple b is $*$-algebraic over some standard $b' \subset b$.

Proof: Suppose $b \subset dcl(a)$ for some standard a. Choose a finite (hence standard) $b' \subset b$ with $a \perp b(b')$. Clearly, b is $*$-algebraic over b'. This proves (2). For (1), notice that when T is ω-stable, then $\mathcal{M}(a/b') = 0$, hence there is some standard $c' \in acl(b')$ such that $tp(a/b'c')$ is stationary. It follows that $b \in dcl(b'c')$. It is easy to replace c' by a finite subset of b so that still $b \in dcl(b'c')$.

$*$-Algebraic elements are very important for the description of m-independence, as the next lemma shows.

LEMMA 1.9: *For any a and A there is $b \in dcl^*(Aa)$ such that b is $*$-algebraic over A and $\mathcal{M}(a/Ab) = 0$. In particular, $\mathcal{M}(b/A) = \mathcal{M}(a/A)$ and for any B with $a \perp B(A)$ we have $a \overset{m}{\perp} B(Ab)$.*

Proof: Say, $a = a_I$. Let I' be the set of A-definable finite equivalence relations $E(x_J, x'_J)$ on the sort of a_J (J varies over finite subsets of I). Let

$$b = \langle a_J/E \colon E = E(x_J, x'_J) \in I' \rangle.$$

Since a, A are $*$-finite, also b is $*$-finite. Clearly, $b \in dcl^*(Aa)$ and b is $*$-algebraic over A. By the choice of b we have $\mathcal{M}(a/Ab) = 0$. We leave the rest of the proof to the reader.

Lemma 1.9 shows that in order to calculate $\mathcal{M}(a/A)$ it suffices to calculate $\mathcal{M}(b/A)$ for some b $*$-algebraic over A. If a is $*$-algebraic over A then for any $B, a \perp B(A)$. Hence by Proposition 1.6 we have that if $a \overset{m}{\perp} a'(A), b$ is $*$-algebraic over A and $\mathcal{M}(b/Aa) = \mathcal{M}(b/Aa') = 0$, then $\mathcal{M}(b/A) = 0$, that is b is algebraic over A. We shall frequently use this remark in the proof of the next theorem. This theorem provides us with an alternative definition of m-normality, parallel to the definition of 1-basedness. Recall that T is 1-based iff for all standard a, b, c there is d with $U(d/ac) = U(d/bc) = 0$ and $a \perp b(cd)$.

THEOREM 1.10: *Assume T is small stable. The following conditions are equivalent.*

1. *T is m-normal.*
2. *For all a, b, c, if $a \perp b(c)$ then for some d with $\mathcal{M}(d/ac) = \mathcal{M}(d/bc) = 0$ and $d \perp ab(c)$, we have $a \overset{m}{\perp} b(cd)$.*
3. *Like (2), but with d additionally $*$-algebraic over c.*

Proof: (3)$\to$(2) is clear. (2) $\to$ (1). Suppose a, b, c, d are as in (2). Let $c' \in acl(c)$ be standard, and such that $tp^*(d/acc'), tp^*(d/bcc')$ are stationary and $\mathrm{Tr}_c(a/bcc'd) = \mathrm{Tr}_c(a/cc'd)$. Since $tp^*(d/bcc')$ is stationary,

$$\mathrm{Tr}_c(a/bcc'd) = \mathrm{Tr}_c(a/bcc') = \mathrm{Tr}_c(a/cc'd).$$

Since $tp^*(d/acc')$ is stationary, $\mathrm{Tr}_c(a/cc'd)$ is $Aut(\mathfrak{C}/acc')$-invariant. Also, $\mathrm{Tr}_c(a/bcc')$ is an open subset of $\mathrm{Tr}_c(a/bc)$ (as $c' \in acl(c)$; here we really do not need the fact that c' is standard). When a, b, c are standard, this argument shows that T is m-normal.

(1) $\to$ (3). Suppose $a \perp b(c)$, and choose standard a', b', c' with $a \subset dcl(a')$, $b \subset dcl(b')$ and $c \subset dcl(c')$. Moreover we can assume that

(a) $c' \perp^{m} ab(c), b' \perp^{m} ac'(b)$ and $a' \perp^{m} b'c'(a)$.

This implies

(b) $a' \perp^{m} b'(ac'), a \perp^{m} b'(bc')$ and $a' \perp b'c'(c)$.

Since T is m-normal, there is $d' \in acl(c')$ such that $\mathrm{Tr}_{c'}(a'/b'c'd')$ is $Aut(\mathfrak{C}/a'c'd')$-invariant. By (b), $\mathrm{Tr}_{c'}(a'/b'c'd')$ is naturally homeomorphic to $\mathrm{Tr}_c(a'/b'c'd')$, which is also $Aut(\mathfrak{C}/a'c'd')$-invariant. Let $I = FE(c)$ and for $E \in I$ let d_E be the name for the set of E-classes meeting $\mathrm{Tr}_c(a'/b'c'd')$. We see that d_E is a standard (imaginary) element, which is $Aut(\mathfrak{C}/a'c'd')$-invariant. Hence $d_E \in dcl(a'c'd') \cap dcl(b'c'd')$. Let $d = \langle d/E, E \in I\rangle$. So d is $*$-finite, $*$-algebraic over c and $d \in dcl^*(a'c'd') \cap dcl^*(b'c'd')$. We have

(c) $\mathcal{M}(d/b'c) = 0 = \mathcal{M}(d/ac')$.

Indeed, $d' \in acl(c')$ gives $\mathcal{M}(d/a'c') = 0 = \mathcal{M}(d/b'c')$. (a) gives that $c' \perp^{m} a'b'(c)$, hence $a' \perp^{m} b'c'(b'c)$. Therefore $\mathrm{Tr}_c(a'/b'c'd')$ (homeomorphic with $\mathrm{Tr}_{b'c}(a'/b'c'd')$) is open in $\mathrm{Tr}_c(a'/b'c)$ (homeomorphic with $\mathrm{Tr}_{b'c}(a'/b'c)$), which means that $d \in acl(b'c)$ and $\mathcal{M}(d/b'c) = 0$. By (b) we have $a' \perp^{m} b'(ac')$, so $\mathcal{M}(d/a'c') = 0 = \mathcal{M}(d/b'c')$ gives $\mathcal{M}(d/ac') = 0$.

Now we have $a \perp^{m} b(cd)$. Indeed, since d is $*$-algebraic over c and $d' \in acl(c')$, by (b) we have $a' \perp b'c'd'(cd)$. Clearly, $\mathrm{Tr}_c(a'/cd) = \mathrm{Tr}_c(a'/b'c'd')$ (as $d' \in acl(c')$). Hence $a' \perp^{m} b'(cd)$ holds. As $a \subset dcl(a')$ and $b \subset dcl(b')$, we get $a \perp^{m} b(cd)$.

To finish, we prove that $\mathcal{M}(d/bc) = 0 = \mathcal{M}(d/ac)$. To see this, notice that $b' \perp^{m} ac'(bc)$, hence using (c) we get $\mathcal{M}(d/bc) = 0$. Similarly, $c' \perp^{m} ab(c)$ gives $b \perp^{m} c'(ac)$ and $\mathcal{M}(d/ac) = 0$.

Regarding condition (3) in Theorem 1.10, notice that if d is $*$-algebraic over c and $\mathcal{M}(d/ac) = 0$ then $d \in acl(ac)$. The next corollary is a kind of coordinatization theorem.

Corollary 1.11: *Assume T is small, stable and m-normal. Then for any a, b with $0 < \mathcal{M}(a/b) < \infty$ there is $a' \in acl(ab)$ such that a' is $*$-algebraic over b*

and $\mathcal{M}(a/a'b) = 1$. In particular, if $\mathcal{M}(a/b) = n$ then there are $*$-algebraic over b $*$-finite tuples $a_0, \ldots, a_{n-1}$ such that $a_i \in acl(a_{i+1}b)$ for $i < n-1$, $a_{n-1} \in acl(ab), \mathcal{M}(a_i/a_{<i}b) = 1$ and $\mathcal{M}(a/a_{<n}) = 0$.

Proof: By Theorem 1.4, $\mathcal{M}(a/b)$ is finite, say n. Choose c with $a \perp c(b)$ and $\mathcal{M}(a/bc) = 1$. We get that the required a' exists by Theorem 1.10. The rest is easy.

$*$-Algebraic tuples may be regarded as types themselves. Specifically, suppose a is $*$-algebraic over b. Choose standard a' with $a \subset dcl(a')$. Hence there is a tuple $f = \langle f_i, i \in I\rangle$ of 0-definable functions such that $a = a_I = f(a')$. Let $p_a = \mathrm{Tr}_b(a'/ab)$. Since p_a is a closed subset of $S(acl(b))$, we can regard it as a type over $acl(b)$ (p_a is just the type over $acl(b)$ generated by $tp(a'/ab)$). We have that $Aut(\mathfrak{C}/ab) = Aut(\mathfrak{C}/p_a b)$. Indeed, first suppose $g \in Aut(\mathfrak{C})$ fixes ab. Then g fixes also $tp(a'/ab)$, hence $g(p_a) = p_a$ and $g \in Aut(\mathfrak{C}/p_a b)$. Now suppose g fixes b and p_a. It follows that $a'' = g(a')$ realizes p_a. But '$a_I = f(x)$' is expressed by a set of formulas contained in $p_a(x)$, hence also $g(a) = f(a'') = a$.

We see that a and p_a are interdefinable in $\mathfrak{C}$ over b. We may use $*$-algebraic tuples to construct models with some special properties. Specifically, we may try to omit the types p_a corresponding to $*$-algebraic tuples a. The choice of the type p_a depends on the choice of standard a' with $a \subset dcl(a')$, so it is not uniquely determined. Thus it is more appropriate to speak of omitting $*$-algebraic tuples rather than the types corresponding to them.

First we must specify when we shall regard a $*$-algebraic tuple realized, or more properly, discerned in a model M. The most liberal approach is that we regard a $*$-algebraic over b tuple a discerned in a model M containing b, if some type p_a corresponding to a is realized in M. In other words, suppose M contains b (if $b = b_J$ is a $*$-finite tuple, then this means that $\{b_i, i \in J\} \subset M$). We say that M **discerns** a over b if for some standard $a' \in M$, $a \subset dcl(a'b)$. Consequently, we say that M omits a over b if M does not discern a over b. We say that M discerns/omits a if M discerns/omits a over $\emptyset$. Notice that omitting a over b is equivalent to omitting potentially many types over $acl(b)$. The following is an omitting $*$-algebraic tuples theorem . It corresponds to omitting $< 2^{\aleph_0}$ complete non-isolated types in the classical case [Sh, IV, 5.16]. The proof is similar.

THEOREM 1.12: *Assume T is small stable. Assume $\kappa < 2^{\aleph_0}, b$ is $*$-finite, $a_\alpha, \alpha < \kappa$, are $*$-algebraic over b, and for all $\alpha < \beta < \kappa, tp^*(a_\alpha/b) = tp^*(a_\beta/b)$ and $\mathcal{M}(a_\alpha/b) > 0$. Then there is a countable model M containing b and omitting every $a_\alpha, \alpha < \kappa$.*

Proof: Extending the signature, we can assume $b = \emptyset$. Let $p = tp^*(a_\alpha/\emptyset), \alpha < \kappa$. Suppose each a_α is indexed by a countable set I. Suppose a' is standard and $a_\alpha \subset dcl(a')$. This means that there is a tuple $f = \langle f_i, i \in I\rangle$ of 0-definable functions such that $a_\alpha = f(a')$. Potentially there may be many such tuples of functions. The next claim shows that we can restrict ourselves to countably many of them.

CLAIM 1: *There are countably many I-indexed tuples $f^n, n < \omega$, of 0-definable functions such, that for any a realizing p and any standard a' with $a \subset dcl(a')$, we have $a = f^n(a')$ for some n.*

Proof: Let a realize p. Since a is $*$-finite and T is small, $S(a)$ is countable. For standard a', the fact that $a \subset dcl(a')$ is witnessed by some I-indexed f depending only on $tp(a'/a)$. Hence there are only countably many cases to consider.

This claim implies also that in every countable model M, only countably many $*$-finite tuples realizing p are discerned. Our aim is to construct now a family of countable models $M_\eta, \eta \in {}^\omega 2$, such that no a realizing p is discerned in more than one of them. We specify M_η by describing the ω-type $p_\eta(x_\omega) = tp(M_\eta/acl(\emptyset))$ in variables $x_\omega = \langle x_n, n < \omega\rangle$. Specifically, we construct a tree of formulas $\{\varphi_\nu(x_\nu), \nu \in {}^{<\omega}2\}$ over $acl(\emptyset)$ such that $x_\nu = \langle x_0, \ldots, x_{n-1}\rangle$, where $n = |\nu|$ and

(a) for every $\eta \in {}^\omega 2$, the set $p_\eta = \{\varphi_{\eta|n}(x_{\eta|n}), n < \omega\}$ is a complete ω-type over $acl(\emptyset)$, whose realization is a model of T,

(b) if $\eta \neq \eta' \in {}^\omega 2$ and $\eta|n \neq \eta'|n$ then whenever $a_{\eta|n}, a_{\eta'|n}$ realize $\bigwedge_{i<n} \varphi_{\eta|i}(x_{\eta|i})$, $\bigwedge_{i<n} \varphi_{\eta'|i}(x_{\eta'|i})$ respectively, and $c \subset a_{\eta|n}, c' \subset a_{\eta'|n}$ then for every $i, j < n, f^i(c) \neq f^j(c')$ (provided c, c' have suitable arity).

The construction relies on the following claim:

CLAIM 2: *Suppose $\varphi(x), \psi(x)$ are consistent formulas over $acl(\emptyset)$ and $j, j' < \omega$. Suppose that for some c realizing φ and c' realizing ψ, $f^j(c), f^{j'}(c')$ realize p. Then there are consistent formulas $\varphi'(x), \psi'(x)$ over $acl(\emptyset)$, below φ, ψ respectively, such that there are no c, c' realizing φ', ψ' respectively, with $f^j(c)$ realizing p and $f^j(c) = f^{j'}(c')$.*

Proof: Say, $\varphi(x), \psi(x)$ are over $d \in acl(\emptyset)$. Choose c, c' realizing φ, ψ respectively, such that $f^j(c), f^{j'}(c')$ realize p. Since $\mathcal{M}(p) > 0$ and $d \in acl(\emptyset)$, we have that $\mathrm{Tr}(f^j(c)/d)$ and $\mathrm{Tr}(f^{j'}(c')/d)$ are infinite. So we can choose a, a' realizing distinct types in $\mathrm{Tr}(f^j(c)/d), \mathrm{Tr}(f^{j'}(c')/d)$ respectively. As $a \neq a'$, there is a finite $J \subset I$ such that $a_J \neq a'_J$. Let $\varphi'(x)$ be $\varphi(x) \wedge f^j_J(x) = a_j$ and $\psi'(x)$ be $\psi(x) \wedge f^{j'}_J(x) = a'_J$. Here $f^j_J = \langle f^j_i, i \in J\rangle$. It suffices to show that φ', ψ' are consistent. Since $a, a', f^j(c), f^{j'}(c')$ realize p, they are $*$-algebraic, hence $a \equiv f^j(c)(d)$ and $a' \equiv f^{j'}(c')(d)$. This implies that there are c^0 realizing φ and c^1 realizing ψ with $f^j(c^0) = a, f^{j'}(c^1) = a'$. Hence φ', ψ' are consistent.

Now for $\eta \in {}^{\omega}2$ let M_η be a countable model, whose universe is an ω-tuple realizing p_η. By Claim 1 and (b), no a realizing p is discerned in more than one of the models $M_\eta, \eta \in {}^{\omega}2$. Hence there is an $\eta \in {}^{\omega}2$ such that M_η omits all $a_\alpha, \alpha < \kappa$.

Remark: In some cases, we can omit simultaneously $2^{\aleph_0}$-many $*$-algebraic types. For instance, suppose a is $*$-algebraic, $a' \subset dcl(a), \mathcal{M}(a'/\emptyset) > 0$ and $\mathcal{M}(a/a') > 0$. By Theorem 1.12, there is a countable model M omitting a'. But any such model must omit also any a'' with $a'' \equiv a(a')$. There are $2^{\aleph_0}$ such a'', as $\mathcal{M}(a/a') > 0$. Notice also that if a is $*$-algebraic and $\mathcal{M}(a/\emptyset) = 0$, then a is discerned in any model M.

2. Traces

In this section we prove that a 1-based superstable theory with few countable models is m-normal. The proof involves an analysis of traces of types under the few models assumption. It turns out that in case of a superstable theory with few countable models these are just finite unions of traces of some formulas, which can be taken weakly normal, if we assume additionally that T is 1-based. When p in the next lemma is a generic type of a group, this lemma is true without the assumption that $\mathcal{M}(p) = 1$ (see the remark after Proposition 3.5). It is not clear if we can omit this assumption in general.

LEMMA 2.1: *Assume T is m-stable, A, B are finite, $p \in S(A)$ (or even $p = tp^*(a/A)$ for some $*$-finite tuple a) and $\mathcal{M}(p) = 1$. Then the set of types $r \in S_{p,nf}(AB)$ such that $\mathcal{M}(r) = 0$ is finite.*

Proof: Suppose not. By Proposition 1.1 we can assume that B is an A-independent set of realizations of p. Moreover, by Lemma 1.9 we can assume that p is $*$-algebraic. In this setting, the failure of the lemma means that the set of a realizing p with $a \in acl(AB)$ is infinite (see the discussion after Proposition 1.6). Since $\mathcal{M}(p) = 1$, acl satisfies the exchange principle on the set of realizations of p. Indeed, the situation resembles much the case of a weakly minimal type, dealt with in [Bu2] (see also [Ne1, lemma 0.2]). The main difference now is that all the stationarizations of p are orthogonal and the set of realizations of p is naturally a topological space (via the identification with $\mathrm{Tr}_A(p)$) homeomorphic with the Cantor set. We can assume that B is *acl*-independent over A and that the size of B is minimal possible.

Let $\mathcal{P}$ denote the set of realizations of p, and in this proof we restrict *acl* to elements of $\mathcal{P}$. Let W be the topological closure of $acl(AB)$ in $\mathcal{P}$.

CASE 1: W is uncountable. Then W has a perfect core W'. W' is $Aut(\mathfrak{C}/AB)$-invariant, hence there is $b \in W'$ with $\mathcal{M}(b/AB) > 0$. Since $\mathcal{M}(b/A) = 1 = \mathcal{M}(p)$, we get $\mathcal{M}(b/AB) = \mathcal{M}(b/A) = 1$, hence the set U of realizations of $tp(b/AB)$ is open in $\mathcal{P}$. Clearly, $U \subset W$. Since $acl(AB)$ is dense in W, it meets U, hence in fact $U \subset acl(AB)$ and $b \in acl(AB)$. But this implies $\mathcal{M}(b/AB) = 0$, a contradiction.

CASE 2: W is countable infinite. This implies easily that $W \subset acl(AB)$ (as for any $b \in W, \mathcal{M}(b/AB) = 0$), hence $acl(AB)$ is topologically closed. Let $b \in acl(AB)$ be an accumulation point. We find an *acl*-basis B' of the set $acl(AB)$, containing b. By the exchange property, $|B| = |B'|$, so we can assume $B = B'$. Let $B'' = B \setminus \{b\}$, and let V be the set of realizations of $tp(b/AB'')$. So V is open in $\mathcal{P}$ and $V \cap acl(AB)$ is infinite and contains b as an accumulation point. It follows that any element of $V \cap acl(AB)$ is an accumulation point of this set, hence $V \cap acl(AB)$ and $acl(AB)$ are uncountable, a contradiction.

The traces of formulas seem least complicated among traces of types. So the next theorem says that under the few-models assumption traces of complete types are not so complicated. More specifically, suppose T is superstable with few countable models and p is a complete type over some finite set A. Then the theorem says that there is a formula φ with $\mathrm{Tr}_A(\varphi) = \mathrm{Tr}_A(p)$. This means that $\varphi(\mathfrak{C})$ is a "large" subset of $p(\mathfrak{C})$. Clearly if p is non-isolated, then necessarily φ forks over A and the parameters of φ are not atomic over A.

THEOREM 2.2: *If T is superstable with few countable models, then for every finite A and $p \in S(A)$ there is a formula φ such that $\mathrm{Tr}_A(\varphi)$ is an open subset of $\mathrm{Tr}_A(p)$. In particular, $\mathrm{Tr}_A(p)$ is the union of traces of some finitely many A-conjugates of φ.*

Proof: If not, choose $p \in S(A)$ for which the theorem fails, with $\mathcal{M}(p) = N$ minimal possible. Clearly $N > 0$, since when $\mathrm{Tr}_A(p)$ is finite, then for any a realizing p the formula $\varphi(x) = (x = a)$ is good.

Choose a non-forking extension $p' \in S(B)$ of p with $\mathcal{M}(p') = N - 1$, for some finite $B \supseteq A$. By the minimality of N there is a formula $\varphi(x, b_0)$ with $\mathrm{Tr}_B(\varphi)$ open in $\mathrm{Tr}_B(p')$. Hence also $\mathrm{Tr}_A(\varphi)$ is open in $\mathrm{Tr}_A(p')$. Further, we could take a non-forking extension p'' of p' over Bb_0 so that $\mathrm{Tr}_A(p'')$ is clopen in $\mathrm{Tr}_A(\varphi(x,b_0))$. So wlog $b_0 \in B$ and $\mathrm{Tr}_A(p') = \mathrm{Tr}_A(\varphi(x,b_0))$. Hence if $s \in S(B) \cap [\varphi(x,b_0)]$ is any type then $\mathrm{Tr}_A(s) = \mathrm{Tr}_A(p')$. So wlog $\varphi(x,b_0)$ isolates a type over Ab_0.

Let $q = tp(b_0/A)$. For any b realizing q let $b^* \in dcl_A(b)$ be the $*$-finite $*$-algebraic over A tuple naming $\mathrm{Tr}_A(\varphi(x,b))$. Specifically, for $E \in FE(A)$ let $\ulcorner\varphi(\mathfrak{C},b)/E\urcorner$ be the name for the finite set of E-classes meeting $\varphi(\mathfrak{C},b)$, and let $b^* = \langle\ulcorner\varphi(\mathfrak{C},b)/E\urcorner\colon E \in FE(A)\rangle$. Also let $q^* = tp(b^*/A)$ and since b^* codes $\mathrm{Tr}_A(\varphi(x,b))$, sometimes we denote $\mathrm{Tr}_A(\varphi(x,b))$ by $X(b^*)$. Notice that

(a) for every b realizing q there is a non-forking extension p'' of p with $\mathcal{M}(p'') = N-1$ and $X(b^*) = \mathrm{Tr}_A(p'')$, and $\mathrm{Tr}_A(p) = \bigcup\{X(b^*)\colon b^* \models q^*\}$.

We can also assume that p, q, φ as above are chosen so as to minimize $\mathcal{M}(q^*)$ (with $N = \mathcal{M}(p)$ fixed).

Claim 1: $\mathcal{M}(q^*) = 1$.

Proof: Since every $X(b^*)$ is nowhere dense in $\mathrm{Tr}_A(p)$, we see that $\mathcal{M}(q^*) > 0$. Suppose $\mathcal{M}(q^*) > 1$. Let q_d^* be a (necessarily non-forking) extension of q^* over Ad (for some finite tuple d) with $\mathcal{M}(q_d^*) = 1$. Wlog b_0^* realizes q_d^*.

Let $X_d = \bigcup\{X(b^*)\colon b^* \models q_d^*\}$. X_d is a closed Ad-invariant subset of $\mathrm{Tr}_A(p)$. Moreover,

(b) X_d is nowhere dense in $\mathrm{Tr}_A(p)$.

Otherwise, we could find $p'' \in S_{p,nf}(Ad)$ with $\mathrm{Tr}_A(p'')$ contained in X_d and open in $\mathrm{Tr}_A(p)$. Then we could essentially replace p by p'' and q^* by q_d^*, contradicting the minimality of $\mathcal{M}(q^*)$.

At this point, since $\mathcal{M}(q_d^*)$ is determined by $\mathrm{Tr}_A(q_d^*)$, we can replace d by the $*$-finite $*$-algebraic over A tuple naming $\mathrm{Tr}_A(q_d^*)$, without violating (b).

Now choose $p'' \in S_{p,nf}(Ad)$ with $\mathrm{Tr}_A(p'')$ being an open subset of X_d. This means that for some standard $d_0 \subseteq d$ and $\delta(x,d_0)$ over Ad_0,

(c) $\mathrm{Tr}_A(p'') = X_d \cap [\delta(x,d_0)] = \bigcup\{X(b^*) \cap [\delta(x,d_0)]\colon b^* \models q_d^*\}$.

Since $\mathcal{M}(X(b^*)) = N-1$ and $X(b^*)$ is a trace of a complete type, we get that $\mathcal{M}(p'') = N-1$ and for every $b^* \models q_d^*$, $X(b^*) \cap [\delta(x,d_0)]$ has non-empty interior in $\mathrm{Tr}_A(p'')$. Just take any $p^* \in S_{p,nf}(Ab^*d)$ with $\mathcal{M}(p^*) = N-1$ and $\mathrm{Tr}_A(p^*) \subseteq X(b^*) \cap [\delta(x,d_0)]$. Necessarily, $\mathrm{Tr}_A(p^*)$ is a clopen subset of both $X(b^*)$ and $\mathrm{Tr}_A(p'')$.

So we find an $E \in FE(A)$ such that for any $b^* \models q_d^*$, for some a with $stp(a/A) \in \mathrm{Tr}_A(p'')$ we have

(d) $X(b^*) \cap [E(x,a)] = \mathrm{Tr}_A(p'') \cap [E(x,a)]$.

For any $b^* \models q_d^*$ let $X'(b^*)$ be the union of all sets of the form $X(b^*) \cap [E(x,a)]$, for which (d) holds. Extending d_0 a little (inside $dcl(Ad) \subseteq acl(A)$) we can assume that $X'(b^*)$ is Ab^*d_0-invariant and is the trace of some formula over Abd_0. So we have that

(e) the set $\{X'(b^*)\colon b^* \models q_d^*\}$ is finite.

Now let $q' = tp(b_0d_0/A)$, let $(b_0d_0)^*$ be the $*$-finite name for $X'(b_0^*)$ and $(q')^* = tp((b_0d_0)^*/A)$.

Clearly $(b_0d_0)^* \in dcl_A(b_0^*d_0)$, however since $\mathcal{M}(q_d^*) > 0$, by (e) we have $\mathcal{M}(b_0^*d_0/A(b_0d_0)^*) > 0$. It follows that $\mathcal{M}((q')^*) < \mathcal{M}(q^*)$, contradicting the minimality of $\mathcal{M}(q^*)$ and proving Claim 1.

Let $f\colon \mathrm{Tr}_A(q) \to \mathrm{Tr}_A(q^*)$ be the continuous mapping sending $stp(b/A)$ to $stp(b^*/A)$.

Claim 2: *$f(\mathrm{Tr}_A(\chi))$ is finite for any formula $\chi(y)$ implying $q(y)$.*

Proof: Suppose not. Say, $\chi(y)$ is over a finite set $D \supseteq A$. Since $f(\mathrm{Tr}_A(\chi))$ is infinite and $\mathcal{M}(q^*) = 1$, by Lemma 2.1, $f(\mathrm{Tr}_A(\chi))$ has non-empty interior in $\mathrm{Tr}_A(q^*)$. Wlog $f(\mathrm{Tr}_A(\chi))$ is a clopen D-invariant subset of $\mathrm{Tr}_A(q^*)$. Let $\varphi'(x) = \exists y(\chi(y) \wedge \varphi(x,y))$. Clearly $\mathrm{Tr}_A(\varphi') = \bigcup\{X(b^*)\colon stp(b^*/A) \in f(\mathrm{Tr}_A(\chi))\}$. Hence finitely many A-conjugates of the set $\mathrm{Tr}_A(\varphi')$ cover $\mathrm{Tr}_A(p)$. It follows that $\mathrm{Tr}_A(\varphi')$ has non-empty interior in $\mathrm{Tr}_A(p)$, contradicting the choice of p.

Now the proof resembles that of [Ne7, theorem 2.4]. We shall construct many countable models of T using just the properties of types q and q^* exhibited in Claims 1 and 2.

Let A' be a finite set containing Ab_0. By smallness choose types $q_n^* \in S(A')$, $n < \omega$, extending q^*, with $\mathrm{Tr}_A(q_n^*)$ disjoint and open in $\mathrm{Tr}_A(q^*)$. Choose $q_n \in S(A')$ extending q with $f(\mathrm{Tr}_A(q_n)) = \mathrm{Tr}_A(q_n^*)$. Since $\mathcal{M}(q_n) \le \mathcal{M}(q) < \omega$, there is a number $K < \omega$ such that for infinitely many n, $\mathcal{M}(q_n) = K$; wlog this holds for all $n < \omega$.

Assume further that A' and the types q_n, q_n^* are chosen so that K is minimal possible. We shall find $n_i, i < \omega$, such that for every $j < \omega$,

(f) if $b_i, i < j$, are A'-independent realizations of $q_{n_i}, i < j$,, then $b_i, i < j$, are m-independent over A'.

We proceed by induction on j. Suppose $n_0, \ldots, n_{j-1}$ have been chosen so that (f) holds. We shall choose n_j. Notice that (f) implies that the type

$$p_{n_0}(x_0) \cup \cdots \cup p_{n_{j-1}}(x_{j-1}) \cup \{\text{“}x_0, \ldots, x_{j-1} \text{ are } A'\text{-independent”}\}$$

has finitely many completions over A', say these are types $tp(c^0/A'), \ldots, tp(c^{k-1}/A')$, where $c^t = \langle c_0^t, \ldots, c_{j-1}^t \rangle$ for some $c_i^t \models p_{n_i}$.

To be able to choose n_j it is enough to show that

(g) for every $t < k$, for at most finitely many $n < \omega$ there is b_n realizing q_n with $b_n \perp c^t(A')$ and $b_n \not\perp^{\mathrm{m}} c^t(A')$.

Suppose this is not true for some $t < k$. Let

$$I = \{n < \omega\colon \text{for some } b_n \models q_n,\ b_n \perp c^t(A') \text{ and } b_n \not\perp^{\mathrm{m}} c^t(A')\}.$$

So I is infinite. For $n \in I$ let $q_n' = tp(b_n/A'c^t)$, $(q_n')^* = tp(b_n^*/A'c^t)$ and since $b_n \not\perp^{\mathrm{m}} c^t(A')$, we can assume that for some $K' < K$, for all $n \in I$, $\mathcal{M}(q_n') = K'$.

By the minimality of K, for almost all $n \in I$, $\mathrm{Tr}_A((q_n')^*)$ is a finite subset of $\mathrm{Tr}_A(q_n^*) \subseteq \mathrm{Tr}_A(q^*)$. We see that the set $\{stp(c^*/A)\colon c^* \models q^* \text{ and } c^* \not\perp^{\mathrm{m}} A'c^t(A)\}$ is infinite, contradicting Lemma 2.1. This proves (g).

Now let $b_i, i < \omega$, be A'-independent realizations of types $q_{n_i}, i < \omega$. By (f), $b_i, i < \omega$, are m-independent over A' (that is, every finite number of them is m-independent over A'). Also, for every $j < \omega$ we have

(h) q_{n_j} is non-isolated over $A'' = A' \cup \{b_i, i \neq j\}$.

Indeed, suppose not. Then there is a formula $\chi(y)$ over a finite subset D of A'' (containing A'), which isolates q_{n_j}. Wlog $\chi(y)$ isolates a type $q' \in S(D)$. So for some non-forking extension $q'' \in S(D)$ of q_{n_j} we have

$$\mathrm{Tr}_A(q'') = \mathrm{Tr}_A(q') = \mathrm{Tr}_A(\chi) \subseteq \mathrm{Tr}_A(q_{n_j}).$$

By (f), for $b'' \models q''$ we have $b'' \perp^{\mathrm{m}} D(A')$, so $\mathrm{Tr}_A(q'')$ is an open subset of $\mathrm{Tr}_A(q_{n_j})$. By Claim 2, $f(\mathrm{Tr}_A(\chi)) = f(\mathrm{Tr}_A(q''))$ is finite. So also $f(\mathrm{Tr}_A(q_{n_j})) = \mathrm{Tr}(q_{n_j}^*)$ is finite (as $\mathrm{Tr}_A(q_{n_j})$ is a finite union of A'-conjugates of $\mathrm{Tr}_A(q'')$), contradicting the choice of $q_n^*, n < \omega$, and proving (h). (In fact, f is an open surjection, as in [Ne5, theorem 0.3].)

By (h), for each set $J \subseteq \omega$ we can construct a countable model M of T containing A' and realizing q_{n_i} iff $i \in J$. This shows that T has many countable models, proving the theorem.

Recall that $\varphi(x, y)$ is weakly normal if for every a, the set $\{\varphi(\mathfrak{C}, b)\colon \varphi(a, b)$ holds$\}$ is finite. In case when T is 1-based we can improve Theorem 2.2.

THEOREM 2.3: *Assume T is 1-based, superstable, with few countbale models, B is finite and $a\perp B$. Then there is a weakly normal formula $\psi(x)$ such that $\mathrm{Tr}(\psi)$ is an open subset of $\mathrm{Tr}(a/B)$.*

Proof: Theorem 2.2 yields a formula $\psi(x,b)$ with $\mathrm{Tr}(\psi)$ being an open subset of $\mathrm{Tr}(a/B)$. By [HP], ψ is a Boolean combination of weakly normal formulas. Possibly diminishing $\psi(\mathfrak{C})$ we can assume that $\psi(x,b)$ is equivalent to $\chi_0(x,b) \wedge \neg\chi_1(x,b) \wedge \cdots \wedge \neg\chi_k(x,b)$, where $\chi_0,\ldots,\chi_k$ are some weakly normal formulas and k is minimal possible. Suppose for a contradiction that $k>0$.

Let $Y = \mathrm{Tr}(\psi(x,b))$ and, for $I \subseteq \{1,\ldots,k\}$, let

$$X_I = \mathrm{Tr}(\chi_0(x,b) \wedge \bigwedge_{i\in I} \neg\chi_i(x,b)).$$

CASE 1: *For some proper subset I of $\{1,\ldots,k\}$, Y is not nowhere dense in X_I.* Wlog $r = stp(a) \in int_{X_I}(Y)$. So choose $E \in FE(\emptyset)$ with $Y \cap [E(x,a)] = X_I \cap [E(x,a)]$, and we can replace $\psi(x,b)$ by $(\chi_0(x,b) \wedge x \in a/E) \wedge \bigwedge_{i\in I} \neg\chi_i(x,b)$, contradicting the minimality of k.

CASE 2: *For every proper subset I of $\{1,\ldots,k\}$, Y is nowhere dense in X_I.* In this case we will show that $\psi(x,b)$ is (equivalent to) a weakly normal formula. Suppose not. Then there are $b_n, n<\omega$, such that $b=b_0$, $b\equiv b_n$, for some a', for all n, $\psi(a',b_n)$ holds and $\psi(x,b_n), n<\omega$, are pairwise non-equivalent.

We will find infinite sets $\omega \supseteq I_0 \supseteq I_1 \supseteq \cdots \supseteq I_k$ and permutation $n_0 = 0, n_1, \ldots, n_k$ of $\{0,1,\ldots,k\}$ such that for every $n,n' \in I_t$, for every $j \le t$, $\chi_{n_j}(x,b_n)$ and $\chi_{n_j}(x,b_{n'})$ are equivalent. In particular, for every $n,n' \in I_k$, $\psi(x,b_n)$ and $\psi(x,b_{n'})$ will be equivalent, a contradiction.

Suppose $t<k$ and we have found I_t and $n_0,\ldots,n_t$; we shall find I_{t+1} and n_{t+1}. Let $n \in I_t$, $J = \{n_1,\ldots,n_t\}$, $Z_J = \mathrm{Tr}(\chi_0(x,b_n) \wedge \bigwedge_{1<j\le t} \neg\chi_{n_j}(x,b_n))$ and $Y_n = \mathrm{Tr}(\psi(x,b_n))$. Since $b_n \equiv b$, by the assumption of case 2 and by the inductive hypothesis, Z_J does not depend on the choice of $n \in I_t$ and for every $n \in I_t$,

(a) Y_n is nowhere dense in Z_J.

Let $Z_n = Z_J \setminus Y_n$. We see that

(b) $Z_n = \{r \in Z_J\colon r(\mathfrak{C}) \cap \chi_0(\mathfrak{C},b_n) \subseteq \bigcup_{j>0} \chi_j(\mathfrak{C},b_n)\}$.

By (a) we can choose $r'' \in \bigcap_{n\in I_t} Z_n$, in particular $r'' \in Z_J$. Let $a'' \in r''(\mathfrak{C})$ realize $\chi_0(x,b_n) \wedge \bigwedge_{1<j\le t} \neg\chi_{n_j}(x,b_n)$ for every $n \in I_t$. By (b), for every $n \in I_t$ there is $j_n > 0$ such that $a'' \in \chi_{j_n}(\mathfrak{C},b_n)$, clearly $j_n \ne n_1,\ldots,n_t$.

By pigeonhole rule for some n_{t+1} there are infinitely many $n \in I_t$ with $j_n = n_{t+1}$. Let $I_{t+1} = \{n \in I_t \colon j_n = n_{t+1}\}$. Clearly this works.

Notice that the proof of this theorem shows that if additionally $T = Th(G)$ for some group G, then the formula $\psi(x)$ may be taken as a coset of an $acl(\emptyset)$-definable subgroup of G.

COROLLARY 2.4: *A 1-based superstable theory with few countable models is m-normal.*

Proof: Suppose $A \subseteq B$ are finite, $a \perp B(A)$, wlog $A = \emptyset$ and let $\varphi(x, b)$ be a weakly normal formula with $\mathrm{Tr}(\varphi(x, b))$ being an open subset of $\mathrm{Tr}(a/B)$. Wlog $\varphi(a, b)$ holds. So $\mathrm{Tr}(\varphi(x, b)) \cap [E(x, a)] = \mathrm{Tr}(a/B) \cap [E(x, a)]$ for some $E \in FE(\emptyset)$. Clearly, $\mathrm{Tr}(\varphi(x, b)) \cap [E(x, a)]$ has finitely many conjugates over a.

PROBLEM: *Find a small superstable theory which is not m-normal.*

QUESTION: *Is any superstable theory with few countable models m-normal?*

The forthcoming paper [Ne8] contains some results suggesting a positive answer to this question. We say that forking is meager on $p \in S(A)$ if for every formula φ forking over A, $\mathrm{Tr}_A(\varphi) \cap \mathrm{Tr}_A(p)$ is nowhere dense in $\mathrm{Tr}_A(p)$. The next corollary improves [Ne7, corollary 2.7(1)], since here we do not assume that p is regular. In fact, this corollary is a restatement of Theorem 2.2.

COROLLARY 2.5: *Assume T is a superstable theory with few countable models, $p \in S(A)$ for some finite $A \subseteq \mathfrak{C}$ and forking is meager on p. Then p is isolated.*

Proof: By Theorem 2.2 there is a formula $\varphi(x)$ with $\mathrm{Tr}_A(\varphi)$ being an open subset of $\mathrm{Tr}_A(p)$. Since forking is meager on p, φ does not fork over A, hence by the open mapping theorem $\mathrm{Tr}_A(\varphi)$ has non-empty interior in $S(acl(A))$. It follows that $\mathrm{Tr}_A(p)$ is open in $S(acl(A))$, hence p is isolated.

3. Generic subgroups of type-definable groups

The examples from the introduction have one feature that we have not explored in the previous section: they are groups. $\mathcal{M}$-rank on generic types of a group has some additional properties worth mentioning. We deal with them in this section.

First we recall some basic notions. We assume $G = (G, \cdot, \dots)$ is a type-definable over $\emptyset$ group in $\mathfrak{C}$. So $G(x)$ is a type over $\emptyset$. $\mathcal{G} \subset S(acl(\emptyset))$ is the set of generic types of G. For any set A, $S_{gen}(A) = \{tp(a/A) \colon a \in G \text{ is generic over } A\}$. We say that a type-definable group $H \leq G$ is generic if $G^0 \leq H$. If a generic subgroup H of G is type-definable over some B, then we can consider $gen(H)$, the set of generic types of H, as a closed subset of both $S(acl(B))$ and of $\mathcal{G}$. If P is a closed subset of $S(acl(A))$ for some finite A then we define $\mathcal{M}(P)$ as the supremum of $\mathcal{M}(p)$, where $p \in S(B)$ for some finite $B \supset A$, p does not fork over A and $\mathrm{Tr}_A(p) \subset P$.

Following [Ne2], for $p, q \in S(acl(\emptyset)) \cap [G(x)], p * q = stp(a \cdot b)$, where a, b are independent realizations of p, q respectively. For sets $X, Y \subseteq S(acl(\emptyset)) \cap [G(x)]$ we define $X * Y$ as the set $\{p * q : p \in X, q \in Y\}$ (that is, the complex product of X, Y). In [Ne2] we prove that $*$ is continuous on $S(acl(\emptyset)) \cap [G(x)]$ coordinatewise. Unfortunately, in general it is not continuous as a binary function. Below we prove however that $*$ restricted to $\mathcal{G} \times \mathcal{G}$ is continuous. Although this result is not used elsewhere in this paper, we include it here, as it clarifies the picture obtained in [Ne2].

Recall that for a stationary type $p, Cb(p)$ is the definable closure of the set of parameters $a_\varphi(p)$ of canonical φ-definitions δ_φ of p. That is, for any $\varphi(x; y)$ and $c \in \mathfrak{C}$ we have $\varphi(x; c) \in p|c$ iff $\delta_\varphi(c; a_\varphi(p))$ holds. In [Ne2, Ne6] we consider the strong topology on $S(acl(\emptyset))$ generated by the basis of open sets $X_{\varphi,c} = \{p \in S(acl(\emptyset)) : a_\varphi(p) = c\}, \varphi \in L, c \in acl(\emptyset)$.

Lemma 3.1: *$*$ is continuous as a binary function (in the strong topology).*

Proof: Let $p, q \in S(acl(\emptyset)) \cap [G(x)]$ and $\varphi(z; t) \in L$. For any $d \in \mathfrak{C}$ we have (in the Hrushovski's notation):

$$\varphi(z, d) \in (p * q)|d \text{ iff } ((d_q y)\varphi(x \cdot y, d)) \in (p|d)(x) \text{ iff } (d_p x)(d_q y)\varphi(x \cdot y, d) \text{ holds.}$$

In other words,

$$\varphi(z, d) \in (p * q)|d \text{ iff } \delta_{\varphi'}(x, d; a_{\varphi'}(q)) \in (p|d)(x) \text{ iff } \delta_{\varphi''}(d, a_{\varphi'}(q), a_{\varphi''}(p)) \text{ holds,}$$

where $\varphi'(y; x, t) = \varphi(x \cdot y, t)$ (y is the 'free' variable in φ'; x, z are the 'parameter' variables there), and $\varphi''(x; t, y) = \delta_{\varphi'}(x, t; y)$.

Hence $\delta_{\varphi''}(t, a_{\varphi'}(q); a_{\varphi''}(p))$ defines $(p * q)|\varphi$. Consequently,

$$a_\varphi(p * q) \in dcl(a_{\varphi'}(q), a_{\varphi''}(p)).$$

Let $c = a_\varphi(p * q), c' = a_{\varphi'}(q)$ and $c'' = a_{\varphi''}(p)$. We see that whenever $(p', q') \in X_{\varphi'',c''} \times X_{\varphi',c'}$ then $p' * q' \in X_{\varphi,c}$. This proves the lemma.

Corollary 3.2: *$*$ is continuous on $\mathcal{G} \times \mathcal{G}$ (in the usual topology). In particular, if $P \subset \mathcal{G}$ is closed then $P * P$ is closed.*

Proof: As in [Ne6, lemma 2.2] one can prove that on $\mathcal{G}$, the usual topology and the strong topology coincide. This is essentially because all generic types have maximal local ranks, and for every finite set Δ of formulas invariant under translation, $\mathcal{G}|\Delta$ is finite. The last clause follows since the continuous image $P*P$ of the compact set $P \times P$ must be compact.

As in [Ne2], for any $P \subset S(acl(\emptyset)) \cap [G(x)], \langle P \rangle$ denotes the minimal type-definable (necessarily over $acl(\emptyset)$) subgroup H of G such that $P(\mathfrak{C}) \subset H$. p^n denotes $p * \cdots * p$ (n times) and P^n denotes $P * \cdots * P$ (n times). Recall that we say that P is A-invariant if for every $f \in Aut(\mathfrak{C}/A)$, $f(P) = P$. We see that P is A-inariant iff there is a set $P' \subset S(A) \cap [G]$ such that $P = \bigcup\{\mathrm{Tr}(r): r \in P'\}$. Using the smallness of T we can prove the following variant of [Ne2, theorem 2.3].

THEOREM 3.3: *Assume A is finite and $P \subset \mathcal{G}$ is A-invariant. Then for some $p \in S_{gen}(A), \mathrm{Tr}(p)$ is an open subset of $gen(\langle P \rangle)$ and for some n, $gen(\langle P \rangle) = \bigcup_{i<n} P^i$.*

Proof: Let $P' = cl(\bigcup_{i<\omega} P^i)$. By [Ne2, theorem 2.3], P' is the set of generic types of $\langle P \rangle$. Clearly, P^i and P' are A-invariant. Since P' is closed and T is small, for some $r \in S_{gen}(A), \mathrm{Tr}(r)$ is an open subset of P' (this is like in Lemma 0.1(8)). As each P^i is A-invariant, if $\mathrm{Tr}(r)$ meets some P^i, then it is contained in this P^i. As P' is compact and metrizable, there are finitely many $r_0, \ldots, r_{n-1} \in \bigcup_{i<\omega} P^i$ such that $P' = \bigcup_{i<n} r_i * \mathrm{Tr}(r)$. Hence for some n, $P' = \bigcup_{i<n} P^i$.

Assume H is a generic subgroup of G, type-definable over a finite set A. Then regarded as a subset of $\mathcal{G}, gen(H)$ is A-invariant, and we define $\mathcal{M}(H)$ as $\mathcal{M}(gen(H))$. Referring to the examples from the introduction, we see that $\mathcal{M}(G_N) = N$ and $\mathcal{M}(G_\infty) = \infty$. Recall in our context the following lemma from [Ne7] ((4) follows from the additivity of $\mathcal{M}$-rank).

LEMMA 3.4: *Assume A is finite and $X, Y \subset \mathcal{G}$ are closed and A-invariant.*

(1) *For any $p \in \mathcal{G}, \mathcal{M}(p * X) = \mathcal{M}(X)$.*
(2) $\mathcal{M}(X \cup Y) = max(\mathcal{M}(X), \mathcal{M}(Y))$.
(3) *If H is a generic subgroup of G type-definable over A and Z is an open subset of $gen(H)$, then $\mathcal{M}(Z) = \mathcal{M}(H)$.*
(4) $\mathcal{M}(X * Y) \leq \mathcal{M}(X) \oplus \mathcal{M}(Y)$.

We do not know any example of a small superstable theory which is not m-normal; the more so we do not know any group G for which the $\mathcal{M}$-gap conjecture is false. Still, if a group like this exists, the smallness imposes some restrictions on generic subgroups of G.

PROPOSITION 3.5: *Assume A is finite and α is an ordinal or ∞. Then there is a unique type-definable over A generic subgroup $G_{\alpha,A}$ of G such that $\mathcal{M}(G_{\alpha,A}) < \omega^\alpha$ and for every $p \in S_{gen}(A)$ with $\mathcal{M}(p) < \omega^\alpha, \mathrm{Tr}(p) \subset gen(G_{\alpha,A})$.*

Proof: Let $P = \bigcup\{\mathrm{Tr}(p): p \in S_{gen}(A) \text{ and } \mathcal{M}(p) < \omega^\alpha\}$ and let $G_{\alpha,A} = \langle P \rangle$. By Lemma 3.4, for every $p \in S_{gen}(A)$ with $\mathrm{Tr}(p) \subset P^i, \mathcal{M}(p) < \omega^\alpha$, hence $\mathrm{Tr}(p) \subset P$. So $P * P = P$ and $P = gen(G_{\alpha,A})$. By Theorem 3.3 we find a single $p \in S_{gen}(A)$ with $\mathrm{Tr}(p)$ open in $gen(G_{\alpha,A})$. By Lemma 3.4(3), $\mathcal{M}(G_{\alpha,A}) = \mathcal{M}(p) < \omega^\alpha$.

We call the group $G_{\alpha,A}$ from Proposition 3.5 the generic α-component of G over A (this resembles [BL]). Notice that $G_{0,A}$ has finitely many generic types, that is it is connected-by-finite. Hence there are finitely many types $p \in S_{gen}(A)$ of finite multiplicity. Essentially this generalizes [Ne1, lemma 0.2] and [Bu2].

COROLLARY 3.6: *The sequence $G_{\alpha,A}, \alpha \in Ord \cup \{\infty\}$ is non-decreasing, and has finitely many members.*

Proof: The first assertion is immediate. Suppose $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots, n < \omega$, and $G_{\alpha_0,A} \neq G_{\alpha_1,A} \neq \cdots$. Let $\mathcal{G}_i \subset \mathcal{G}$ be $gen(G_{\alpha_i,A})$. We see that $\mathcal{M}(\mathcal{G}_0) < \mathcal{M}(\mathcal{G}_1) < \cdots$. Consider $H = \langle \bigcup_{n<\omega} \mathcal{G}_n \rangle$. By Theorem 3.3, for some $p \in S_{gen}(A), \mathrm{Tr}(p)$ is open in $gen(H)$. Hence by Lemma 3.4(3), $\mathcal{M}(p) = \mathcal{M}(H)$. On the other hand, $\mathrm{Tr}(p)$ meets some $\mathcal{G}_n$, hence $\mathrm{Tr}(p) \subset \mathcal{G}_n$ (as $\mathcal{G}_n$ is A-invariant). Thus $\mathcal{M}(p) \leq \mathcal{M}(\mathcal{G}_n) < \mathcal{M}(\mathcal{G}_{n+1}) \leq \mathcal{M}(H) = \mathcal{M}(p)$, a contradiction.

More generally one can prove that the set $\{\alpha$: α is the exponent of the leading term of $\mathcal{M}(H)$ for some type-definable over A subgroup H of $G\}$ is finite. For instance, if G is G_∞ from the introduction, then for any finite $A, G_{\infty,A} = G_{0,A}$. The most promising candidate for a counterexample to the $\mathcal{M}$-gap conjecture seems an expansion by constants (algebraic over $\emptyset$) of some weakly minimal group G, and the main obstacle is preservation of smallness. Proposition 3.5 and Corollary 3.6 indicate that any attempt to find, say, a weakly minimal group G being a counterexample to the $\mathcal{M}$-gap conjecture would still have to deal with some generic subgroups of G. On the other hand (in order to avoid m-normality), in such a G there would have to be a generic type p over some finite set, with trace not being a Boolean combination of $*$-translations of traces of generic type-definable over $\emptyset$ subgroups of G.

We will conclude this paper comparing "measure and category in superstable theories". One can define a notion of independence intermediate between m-independence and forking independence. This is done essentially in [LS]. Specifically, suppose A is a finite set of parameters and $p \in S(A)$. We can define probabilistic Haar measure μ_p on $S(acl(A))$ concentrated on $\mathrm{Tr}_A(p)$, such that for any $E \in FE(A)$ and a realizing p, $\mu_p(\mathrm{Tr}_A(p) \cap [E(x,a)]) = 1/n$, where n is the number of E-classes on $p(\mathfrak{C})$; $\mu_p(S(acl(A)) \setminus \mathrm{Tr}_A(p)) = 0$.

Let $\mu_A = \sum_{p \in S(A)} \mu_p$. We define μ-independence (μ stands for measure) as follows.

For any finite a, b,

$$a \overset{\mu}{\perp} b(A) \text{ iff } a \perp b(A) \text{ and } \mu_A(\mathrm{Tr}_A(a/Ab)) > 0.$$

Since any open subset of $\mathrm{Tr}_A(p)$ (where $p \in S(A)$) has positive measure, we have that

$$\overset{m}{\perp} \Rightarrow \overset{\mu}{\perp} \Rightarrow \perp.$$

Also, just as in case of $\overset{m}{\perp}$ and $\perp$ we have that $\overset{\mu}{\perp}$ is symmetric and transitive [LS, lemmas 2.2, 2.4]. Also, since $\overset{m}{\perp}$ implies $\overset{\mu}{\perp}$, extension property for $\overset{m}{\perp}$ (Lemma 0.1(7)) implies extension property for $\overset{\mu}{\perp}$.

However, the next proposition and corollary, proved by Predrag Tanovic, show that in some important cases m-independence and μ-independence coincide.

PROPOSITION 3.7 (Predrag Tanovic): *If T is small, stable and m-stable then μ-independence equals m-independence.*

Proof: If not, then for some finite A and $p \in S(A)$, for some b and $q = q(x,b) \in S(Ab)$, which is a non-forking extension of p, we have that $\mu_p(\mathrm{Tr}_A(q)) = \alpha > 0$ and $\mathrm{Tr}_A(q)$ is nowhere dense in $\mathrm{Tr}_A(p)$. We can choose q so as to minimize $\mathcal{M}(q)$.

For $b' \equiv b(A)$ let $q(x,b')$ be the A-conjugate of $q(x,b)$. If additionally $b \overset{m}{\perp} b'(A)$ then

(a) $\mathrm{Tr}_A(q(x,b)) \cap \mathrm{Tr}_A(q(x,b'))$ is nowhere dense in $\mathrm{Tr}_A(q(x,b'))$.

Indeed, otherwise $\mathrm{Tr}_A(q(x,b)) \cap \mathrm{Tr}_A(q(x,b'))$ has non-empty interior in $\mathrm{Tr}_A(q(x,b'))$, so for some a realizing $q(x,b) \cup q(x,b')$ we have $a \overset{m}{\perp} b(Ab')$. $b \overset{m}{\perp} b'(A)$ and transitivity of m-independence give $a \overset{m}{\perp} b(A)$, contradicting that $\mathrm{Tr}_A(q)$ is nowhere dense in $\mathrm{Tr}_A(p)$.

It follows that for every $r \in S(Abb')$ which is a non-forking extension of $q(x,b) \cup q(x,b')$, we have $\mathcal{M}(r) < \mathcal{M}(q)$, so by the minimality of $\mathcal{M}$-rank of q, $\mu_p(\mathrm{Tr}_A(r)) = 0$. Hence also

(b) $\mu_p(\mathrm{Tr}_A(q(x,b)) \cap \mathrm{Tr}_A(q(x,b'))) = 0$.

Now let $b_n, n < \omega$, be m-independent over A, with $b_n \equiv b(A)$. By (b) we see that $\mathrm{Tr}_A(q(x,b_n)), n < \omega$, is a family of measure-disjoint subsets of $\mathrm{Tr}_A(p)$, of measure $\alpha > 0$, a contradiction.

Since any superstable theory with few countable models is m-stable we get the following corollary.

COROLLARY 3.8: *If T is superstable with few countable models then m-independence equals μ-independence.*

The example from Section 0 shows that not all m-normal theories are m-stable. However for such theories μ-independence also equals m-independence.

PROPOSITION 3.9: *If T is small, stable and m-normal, then μ-independence equals m-independence.*

Proof: If not, choose p, q as in the proof of Proposition 3.7 (without minimizing $\mathcal{M}(q)$). Let a realize q. By Remark 0.2 choose $c \in acl(A)$ such that $X = \mathrm{Tr}_A(a/Abc)$ is $Aut(\mathfrak{C}/Aca)$-invariant. Since μ_p is $Aut(\mathfrak{C}/A)$-invariant and X is open in $\mathrm{Tr}_A(q)$, we have that $\alpha = \mu_p(X) > 0$. Similarly, letting Y denote $\mathrm{Tr}_A(a/Ac)$, we have that Y is open in $\mathrm{Tr}_A(p)$ and X is nowhere dense in Y. Notice that

(a) whenever $a' \equiv a(Ac)$ and X' is the corresponding conjugate of X over Ac then X' is $Aut(\mathfrak{C}/Aca')$-invariant.

Since $X = \mathrm{Tr}_A(a/Abc)$, we have also that

(b) whenever $stp(a'/A) \in X$, then X is $Aut(\mathfrak{C}/Aca')$-invariant and a and a' are conjugate over $Ac\{X\}$.

Let $\mathcal{X}$ be the family of all conjugates of X over Ac. We see that $\mathcal{X}$ is an infinite family of closed nowhere dense sets of measure α covering Y. By (a) and (b) all sets in $\mathcal{X}$ are pairwise disjoint, a contradiction.

Added in proof: Recently I proved that any superstable theory with few countable models is m-normal, answering positively a question from Section 2.

References

[Ba] J. T. Baldwin, *Fundamentals of Stability Theory*, Springer, Berlin, 1988.

[BL] Ch. Berline and D. Lascar, *Superstable groups*, Annals of Pure and Applied Logic **30** (1986), 1–45.

[Bu1] S. Buechler, *The classification of small weakly minimal sets I*, in *Classification Theory, Chicago 1985, Proceedings* (J. T. Baldwin, ed.), Springer, Berlin, 1987, pp. 32–71.

[Bu2] S. Buechler, *Classification of small w.m. sets II*, Journal of Symbolic Logic **53** (1988), 625–635.

[HP] E. Hrushovski and A. Pillay, *Weakly normal groups*, in *Logic Colloquium '85* (The Paris Logic Group, ed.), North-Holland, Amsterdam, 1987, pp. 233–244.

[LS] M. C. Laskowski and S. Shelah, *Forcing isomorphism II*, Journal of Symbolic Logic **61** (1996), 1305–1320.

[M] M. Makkai, *A survey of basic stability theory, with particular emphasis on orthogonality and regular types*, Israel Journal of Mathematics **49** (1984), 181–238.

[Ne1] L. Newelski, *A proof of Saffe's conjecture*, Fundamenta Mathematicae **134** (1990), 143–155.

[Ne2] L. Newelski, *On type definable subgroups of a stable group*, Notre Dame Journal of Formal Logic **32** (1991), 173–187.

[Ne3] L. Newelski, *A model and its subset*, Journal of Symbolic Logic **57** (1992), 644–658.

[Ne4] L. Newelski, *Meager forking*, Annals of Pure and Applied Logic **70** (1994), 141–175.

[Ne5] L. Newelski, *$\mathcal{M}$-rank and meager types*, Fundamenta Mathematicae **146** (1995), 121–139.

[Ne6] L. Newelski, *On atomic or saturated sets*, Journal of Symbolic Logic **61** (1996), 318–333.

[Ne7] L. Newelski, *$\mathcal{M}$-rank and meager groups*, Fundamenta Mathematicae **150** (1996), 149–171.

[Ne8] L. Newelski, *Geometry of $*$-finite types*, Journal of Symbolic Logic, to appear.

[Sh] S. Shelah, *Classification Theory*, 2nd edition, North-Holland, Amsterdam, 1990.

[SHM] S. Shelah, L. Harington and M. Makkai, *A proof of Vaught's conjecture for ω-stable theories*, Israel Journal of Mathematics **49** (1984), 259–278.